

# Ultracold lattice gases with periodically modulated interactions

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We show that a time-dependent magnetic field inducing a periodically modulated scattering length may lead to interesting novel scenarios for cold gases in optical lattices, characterized by a nonlinear hopping depending on the number difference at neighboring sites. We discuss the rich physics introduced by this hopping, including pair superfluidity, exactly defect-free Mott-insulator states for finite hopping, and pure holon- and doublon superfluids. We also address experimental detection, showing that the introduced non-linear hopping may lead in harmonically trapped gases to abrupt drops in the density profile marking the interface between different superfluid regions.

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Ultracold atoms in optical lattices formed by laser beams provide an excellent environment for studying lattice models of general relevance in condensed-matter physics, and in particular, variations of the celebrated Hubbard model [1, 2]. Cold lattice gases allow for an unprecedented degree of control of various experimental parameters, even in real time. In particular, interparticle interactions can be changed by means of Feshbach resonances [3]. Moreover, recent milestone achievements allow for site-resolved detection, permitting the study of in-situ densities [4, 5], and more involved measurements, as that of non-local parity order [6].

The modulation of the lattice parameters in real time opens interesting possibilities of control and quantum engineering. In particular, a periodic lattice modulation translates by means of Floquet theorem [7] into a modified hopping constant [8], which may even reverse its sign as shown in experiments [9, 10]. This technique has been employed to drive the Mott-insulator (MI) to superfluid (SF) transition [11], and to simulate frustrated classical magnetism [12]. Recent experiments have explored as well the fascinating perspectives offered by periodically driven lattices in strongly correlated gases [13, 14].

The effective Hubbard-like models describing these ultracold lattice gases are typically characterized by a hopping rate which is independent of the number of particles at the sites. This is, however, not necessarily the case. Multiband physics may lead to occupation-dependent hopping [15, 16]. In addition, long-range dipole-dipole interactions may lead to number-dependent hoppings as well, for sufficiently large dipole strengths [17]. A major consequence of non-linear hopping is the possibility to observe pair superfluidity (PSF) [17, 18].

In this Letter, we consider a cold lattice gas in the presence of a periodically modulated magnetic field. In the vicinity of a Feshbach resonance, this field induces modulated interparticle interactions. Interestingly, Ref. [19] has shown that such periodic modulations of the interaction potential may lead to a many-body coherent destruction of tunneling in two-mode BECs. As shown below, the generalization of this effect to lattice gases,

leads under proper conditions to an effective Hubbard-like model with a non-linear hopping which, in contrast to other proposals mentioned above, depends on the difference of occupations at neighboring sites, and retains its non-linear character even for weak lattices. We discuss the rich physics introduced by this hopping, including PSF phases, exactly defect-free MI states for finite hopping, and pure holon- and doublon superfluids. We also address experimental detection, showing that the studied non-linear hopping may lead to abrupt drops in the density profile of harmonically trapped gases.

We consider bosons in a lattice in the presence of a periodically modulated magnetic field  $B(t) = B(t + T)$  (with period  $T = 2\pi/\omega$ ) chosen close to a Feshbach resonance, where the  $s$ -wave scattering length acquires the form  $a(t) = a_{bg} \left(1 + \frac{\Delta B}{B(t) - B_r}\right) = a_0 + \sum_{l>0} a_l \cos(l\omega t)$ . Here  $\Delta B$  and  $B_r$  determine the width and position of the resonance, respectively, and  $a_{bg}$  is the background scattering length [3]. Assuming that the gap between the first two lattice bands is much larger than any other energy scale in the problem, we consider only the lowest band and describe the system by a Bose-Hubbard model (BHM) [1, 2]:

$$H(t) = -J \sum_{\langle ij \rangle} b_i^\dagger b_j + \frac{U(t)}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) - \sum_i \mu \hat{n}_i, \quad (1)$$

where  $b_i$  ( $b_i^\dagger$ ) is the bosonic annihilation (creation) operator at site  $i$ ,  $\hat{n}_i = b_i^\dagger b_i$ ,  $\mu$  is the chemical potential,  $J > 0$  is the hopping rate and  $\langle \dots \rangle$  denotes nearest neighbors. Interactions are characterized by a coupling  $U(t) = U_0 + \sum_{l>0} U_l \cos(l\omega t) = U_0 + \tilde{U}(t)$ , with  $U_0 > 0$  and  $U_l = \frac{4\pi\hbar^2 a_l}{M} \int d^3r |w(\mathbf{r})|^4$ . Here  $w(\mathbf{r})$  is the lowest Wannier function and  $M$  is the atomic mass.

We apply a similar analysis as the one used for shaken lattices [8]. We specify a Floquet basis

$$|\{n_j\}, m\rangle = e^{im\omega t} e^{-i\frac{V(t)}{2} \sum_j \hat{n}_j (\hat{n}_j - 1)} |\{n_j\}\rangle, \quad (2)$$

where  $m$  defines the Floquet sectors and  $|\{n_j\}\rangle$  is the Fock basis, characterized by the atom number at each

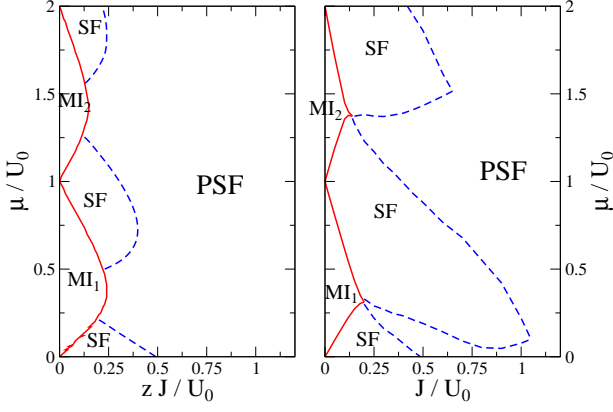


FIG. 1: (Color online) Phase diagram for  $\Omega = 4$  using GA (left) and DMRG (right). Solid curves define the MI lobes, whereas dashed curves are the SF-PSF boundaries.

site. In Eq. (2), we defined  $V(t) = \int^t \tilde{U}(t') dt' / \hbar$ . We introduce the time-averaged scalar product  $\langle \langle \{n'_j\}, m' | \dots | \{n_j\}, m \rangle \rangle = \frac{1}{T} \int_0^T \langle \{n'_j\}, m' | \dots | \{n_j\}, m \rangle dt$ , and establish the matrix elements:

$$\begin{aligned} & \langle \langle \{n'_j\}, m' | [H(t) - i\hbar\partial_t] | \{n_j\}, m \rangle \rangle \\ &= \delta_{m,m'} \langle \langle \{n'_j\} | H_m | \{n_j\} \rangle \rangle \\ & - J \sum_{\langle i,j \rangle} \langle \langle \{n'_j\} | b_i^\dagger F_{m'-m} (\hat{n}_i - \hat{n}_j) b_j | \{n_j\} \rangle \rangle, \end{aligned} \quad (3)$$

with  $H_m = m\hbar\omega + \frac{U_0}{2} \sum_j \hat{n}_j (\hat{n}_j - 1) - \sum_j \mu \hat{n}_j$  and  $F_m(x) = \frac{1}{T} \int_0^T dt e^{-imt} e^{iV(t)x}$ . If  $\hbar\omega \gg J, U_0$ , we may restrict to a single Floquet sector  $m = 0$ , resulting in an effective time-independent Hamiltonian of the form

$$\begin{aligned} H_{\text{eff}} &= -J \sum_{\langle i,j \rangle} b_i^\dagger F_0 (\hat{n}_i - \hat{n}_j) b_j \\ &+ \frac{U_0}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) - \mu \sum_i \hat{n}_i. \end{aligned} \quad (4)$$

Hence, interactions with a periodic modulation result in a non-linear hopping term, which depends on the atom number difference between neighboring sites. Note that this non-linear character remains relevant for any value of the bare hopping  $J$ . In the following we discuss the specific case  $\tilde{U}(t) = U_1 \cos \omega t$ . In this case,  $F_0(x) = \mathcal{J}_0(\Omega x)$ , with  $\mathcal{J}_0$  the Bessel function and  $\Omega = U_1/\hbar\omega$ , generalizing the result of Ref. [19] for two-well BECs.

Insight on Eq. (4) is gained by means of a Gutzwiller Ansatz (GA) for the ground state [20],  $|G\rangle = \prod_j \sum_n f_n(j) |n_j\rangle$ , where  $f_n(j)$  are variational parameters ( $\sum_n |f_n(j)|^2 = 1$ ), determined by minimizing  $\langle G | H_{\text{eff}} | G \rangle$ . Results by choosing homogeneous real  $f_n(j) = f_n$  [21] are shown in Fig. 1 (left), where we depict the mean-field phase diagram for  $\Omega = 4$  ( $\mathcal{J}_0(\Omega) \simeq -0.4$ ) as a function of  $\mu/U_0$  and  $zJ/U_0$ , with  $z$  the coordination number. As usual, MI phases are characterized

by integer  $\langle \hat{n}_i \rangle$ , and vanishing single-particle- and pair-condensation parameters,  $\langle b_i \rangle$  and  $\langle b_i^2 \rangle$ . In addition to the usual SF phase, we find a PSF phase characterized by  $|\langle b_i^2 \rangle| > |\langle b_i \rangle|^2$ . Pair superfluidity is especially pronounced in the vicinity of integer  $\langle \hat{n}_i \rangle$ . Our GA results show that PSF phases only occur if  $\mathcal{J}_0(\Omega) < 0$ . This may be understood by considering integer filling  $\langle \hat{n}_i \rangle = n$ , and restricting the variational space to  $f_{n\pm 1} = \frac{\sin \eta}{\sqrt{2}} e^{i\varphi_{\pm}}$  and  $f_n = \cos \eta$ . For  $\mathcal{J}_0(\Omega) < 0$ , energy minimization gives  $\Delta\varphi \equiv \varphi_+ - \varphi_- = \pi$ , while  $\Delta\varphi = 0$  for  $\mathcal{J}_0(\Omega) > 0$ . As a result, for dominant hopping ( $2Jz/U_0 \gg 1$ ), PSF needs  $4\sqrt{n(n+1)} > (\sqrt{n} + \text{sign}(\mathcal{J}_0(\Omega))\sqrt{n+1})^2$ , which can only be fulfilled if  $\mathcal{J}_0(\Omega) < 0$ .

To complement the mean-field GA results, we have also employed numerically exact methods in 1D. In particular, we used the density-matrix renormalization group (DMRG) [22] with up to 40 sites and keeping 200 states, and a related method, the infinite time-evolving block decimation (iTEBD) [23] using a Schmidt dimension of 200. We have monitored the behavior of single-particle and pair correlations,  $G_1(i, j) \equiv \langle b_i^\dagger b_j \rangle$  and  $G_2(i, j) \equiv \langle (b_i^\dagger)^2 b_j^2 \rangle$ , respectively. Both decay exponentially in the MI. PSF phases are characterized by a dominant  $G_2$ , i.e., a slower power-law decay than  $G_1$ . The opposite characterizes the SF phase. Figure 1 (right) shows the 1D phase diagram for  $\Omega = 4$ , which closely resembles the GA one. Similar to the GA, we observe a PSF phase, which for integer  $\langle \hat{n} \rangle$  approaches all the way to the tip of the MI lobes. On the other hand, away from the lobe tips we observe a direct MI-SF transition. Our 1D results also confirm the absence of PSF for  $\mathcal{J}_0(\Omega) > 0$ .

The case  $\mathcal{J}_0(\Omega) = 0$  is particularly interesting, since for neighboring sites  $i$  and  $j$  with equal number of particles, the process  $|n\rangle_i |n\rangle_j \rightarrow |n \pm 1\rangle_i |n \mp 1\rangle_j$  is forbidden. However, the hopping  $|n \pm 1\rangle_i |n\rangle_j \rightarrow |n\rangle_i |n \pm 1\rangle_j$  is still characterized by the usual rate  $J$ . This difference has a remarkable impact for both the MI- and the SF phases.

For  $J = 0$ , the ground state of Eq. (4) is, as for the standard BHM ( $\Omega = 0$ ), a defect-free MI  $\bigotimes_j |n_j\rangle$  for  $n - 1 < \mu/U_0 < n$  [24]. For  $\Omega = 0$  and  $J > 0$ , this state is not an eigenstate of Eq. (4), and quantum fluctuations induce a finite particle-hole population in the MI with an associated non-local parity order [6]. Interestingly, the defect-free state remains an eigenstate of (4) for  $\mathcal{J}_0(\Omega) = 0$ . As a result, the whole MI lobe is characterized by the absence of particle-hole defects. Although this is typically an artifact in the mean-field GA, in this case it is an exact result, and indeed our 1D iTEBD results show a vanishing variance  $(\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2$  within the Mott region (see Fig. 2).

Conversely, particles or holes ( $|n \pm 1\rangle$ ) on top of the state  $\bigotimes_j |n_j\rangle$  acquire also remarkable properties. For any  $\Omega$ , extra particles and holes move with a hopping rate  $(n+1)J$  and  $nJ$ , respectively. For  $\Omega = 0$  defects are unstable, being created and destroyed by processes

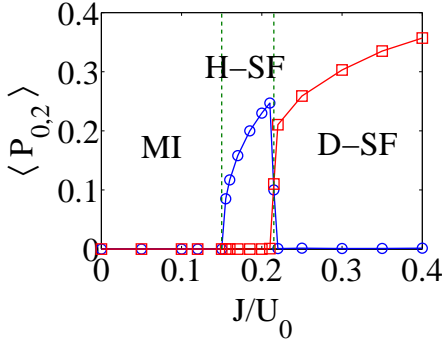


FIG. 2: (Color online) iTEBD results for the holon (circles) and doublon (squares) populations as a function of  $J/U_0$  for a 1D system with  $\mu/U_0 = 0.3$  and  $\Omega = 2.405$ . Note the absence of defects in the MI ( $J/U_0 < 0.15$ ), and the appearance of the holon (H-SF) and doublon SF (D-SF).

$|n\rangle_i |n\rangle_j \leftrightarrow |n \pm 1\rangle_i |n \mp 1\rangle_j$ . Since these processes are forbidden for  $\mathcal{J}_0(\Omega) = 0$ , defects remain stable [25]. Neglecting occupations other than  $n$  and  $n \pm 1$ , the defects are described by an effective Hamiltonian  $H_h + H_p$ , where

$$H_h = -Jn \sum_{\langle i,j \rangle} h_i^\dagger h_j + (\mu - U_0(n-1)) \sum_i h_i^\dagger h_i, \quad (5)$$

$$H_p = -J(n+1) \sum_{\langle i,j \rangle} p_i^\dagger p_j + (U_0 n - \mu) \sum_i p_i^\dagger p_i, \quad (6)$$

characterize, respectively, the physics of holes and particles, with the hard-core assumption  $p_i^\dagger p_i + h_i^\dagger h_i = 0$  or 1, with  $h_i$  ( $p_i$ ) the bosonic operators for extra holes (particles) at site  $i$ . In Eqns. (5) and (6) we have set the energy of the defect-free MI state  $E_{\text{MI}} = 0$ . Thus the system behaves as a two-component hard-core lattice Bose gas. For higher dimensions, a dilute gas of extra holes (holon gas) may be considered as a basically free (superfluid) Bose gas, with a dispersion  $E_h(\mathbf{q}) = \mu - U_0(n-1) + n\epsilon_{\mathbf{q}}^0$ , where  $\epsilon_{\mathbf{q}}^0 = -2J \sum_{j=x,y,z} \cos(q_j d)$  for a 3D cubic lattice and  $d$  is the lattice spacing. On the other hand, the dilute gas of extra particles (“doublon” gas [26]) has a dispersion  $E_p(\mathbf{q}) = U_0 n - \mu + (n+1)\epsilon_{\mathbf{q}}^0$ .

At zero temperature, the defect gas condenses for  $\mu < \mu_c \equiv U_0(n-1/2) - Jz$  at the bottom of the holon band,  $E_h(0)$ , acquiring a pure holon character. On the other hand, for  $\mu > \mu_c$  the system condenses at  $E_p(0)$  into a pure doublon gas. Hence, remarkably, we expect an abrupt jump of  $\langle \hat{n} \rangle$  (i.e. a diverging compressibility) at the line  $\mu = \mu_c$ , which coincides with the line of integer  $\langle \hat{n} \rangle = n$ . Figure 3 depicts our GA results for the density as a function of  $\mu/U_0$  and  $J/U_0$ , which, as expected from the previous discussion, presents an abrupt jump between a holon and a doublon superfluid.

In 1D, the defects behave, due to the hard-core constraint, rather as a two-component Tonks gas, but a similar two-band reasoning as above applies, and we may also expect the existence of pure holon and doublon superfluids. Figure 2 shows our iTEBD results in the vicinity of

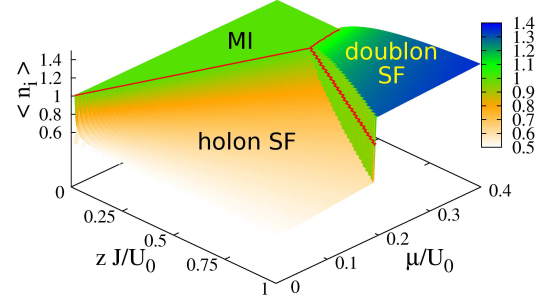


FIG. 3: (Color online) Homogeneous GA results for  $\langle \hat{n} \rangle$  as a function of  $J/U_0$  and  $\mu/U_0$  for  $\Omega = 2.405$ . Red curves denote the boundary of the MI and the line of integer filling 1. Note the abrupt jump in the density at that line, indicating the transition between the holon SF and doublon SF regimes.

$\langle \hat{n} \rangle = 1$  for the holon (doublon) populations  $\langle \hat{P}_0 \rangle$  ( $\langle \hat{P}_2 \rangle$ ), with  $\hat{P}_n = \prod_{n' \neq n} (\hat{n} - n') / (n - n')$ . In addition to the MI phase characterized by  $\langle \hat{P}_0 \rangle = \langle \hat{P}_2 \rangle = 0$ , we observe a holon-SF ( $\langle \hat{P}_2 \rangle = 0$ ) and an abrupt jump to a doublon-SF ( $\langle \hat{P}_0 \rangle = 0$ ). Note that pure doublon- or holon-SF exclude PSF.

At constant  $\mu$  the system undergoes a MI – doublon (holon) SF transition at a critical tunneling  $J_c(\mu)$  for which  $E_{p(h)}(0) = E_{\text{MI}}$ . In 1D, our iTEBD results show that this transition retains a commensurate/incommensurate nature as in the usual 1D BHM [27], characterized by  $\langle \hat{P}_{0,2} \rangle \sim \sqrt{J - J_c}$ . This growth is illustrated for the Mott – holon-SF transition in Fig. 2. On the other hand, at constant integer  $\langle \hat{n} \rangle$ , there is no 1D MI-SF transition at finite hopping  $J$ . This result, expected from the theory of two-component Tonks gases [28], is due to the absence of processes  $|n\rangle_i |n\rangle_j \leftrightarrow |n \pm 1\rangle_i |n \mp 1\rangle_j$  which precludes that doublons and holons can swap their positions through second-order super-exchanges. As a result, if holons and doublons coexist, (which only happens at the singular integer filling line) superfluidity is absent. Our DMRG results for  $\langle \hat{n} \rangle = 1$  confirm indeed that  $G_{1,2}$  decay exponentially for any finite  $J$ . Interestingly, there is however a clear transition between a defect-free insulator and an insulator with a finite density of holon-doublon pairs.

For a finite but small  $\mathcal{J}_0(\Omega)$ , the SF regions retain to a large extent their holon/doublon character, although the concentration of doublons/holons in the holon/doublon SF increases for growing  $\mathcal{J}_0(\Omega)$  and  $J$ . The coexistence region for holons and doublons is hence not any more singular, although it remains characterized by a large compressibility for small  $\mathcal{J}_0(\Omega)$ . For  $\mathcal{J}_0(\Omega) < 0$  this coexistence region becomes the PSF phase discussed above. Away from the Mott-tip a direct MI-SF transition is observed, as discussed above, since at the MI boundary holons and doublons do not coexist.

Let us finally discuss some experimental questions. Optimal experimental conditions for periodically mod-

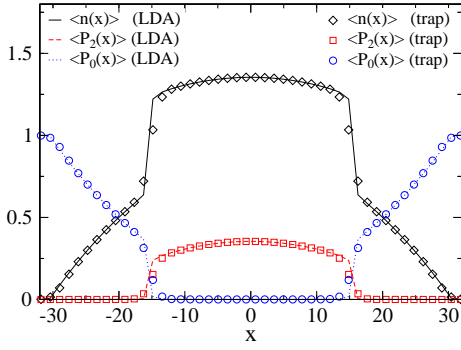


FIG. 4: (Color online) GA results for the site densities  $\langle \hat{n}_j \rangle$  and the on-site holon and doublon populations for a 2D lattice with a harmonic confinement  $V(j_x, j_y) = V_0(j_x^2 + j_y^2)$ , with  $V_0/J = 0.0075$ , interaction  $U_0/J = 5.33$ , a central chemical potential  $\mu_0/J = 3$ , and  $\Omega = 2.45$  ( $\mathcal{J}_0(\Omega) \simeq 0$ ). Note the central doublon-SF region, surrounded by a holon-SF ring, and the abrupt density drop separating both regimes. Lines indicate local-density approximation (LDA) results.

ulated interactions are provided by  $^{85}\text{Rb}$ , which has a particularly large  $a_{bg} \simeq -400a_B$  (with  $a_B$  the Bohr radius), and a broad Feshbach resonance at  $B_r = 155.2\text{G}$ , with a width  $\Delta B = -11.6\text{G}$  [29]. The desired form  $a(t) \approx a_0 + a_1 \cos(\omega t)$  can be achieved for a magnetic field dependence  $B(t)/\text{G} \simeq 167.56 + 5.58 \cos(\omega t)$ , with  $a_0 \approx 20a_B$  and  $a_1 \approx 200a_B$ . We consider a lattice spacing  $d = 0.5\mu\text{m}$ , and potential depth  $V_L = sE_R$ , where  $E_R = \hbar^2 \pi^2 / 2md^2$  is the recoil energy. For  $s \approx 17$  ( $J \ll U_0$ ), the value  $\Omega = 2.4$  ( $\mathcal{J}_0(\Omega) \simeq 0$ ) is obtained for  $\omega \simeq 2\pi \times 900\text{ Hz} \gg U_0/\hbar = 2\pi \times 217\text{ Hz}$ , ensuring that only one Floquet manifold is relevant.

In order to address the question of detection, we have to consider the transformation (2) between the Floquet  $|\{n_j\}, m\rangle$  and the Fock  $|\{n_j\}\rangle$  basis. The densities  $\langle \hat{n}_i \rangle$  are equivalent in both, therefore the large compressibility regions characteristic of  $|\mathcal{J}_0(\Omega)| \simeq 0$  may be revealed in in-situ experiments with an additional harmonic confinement. This is illustrated in Fig. 4, where we show inhomogeneous GA results for a harmonic trap in 2D. As expected from local-density approximation, we observe an abrupt jump in the  $\langle \hat{n}_i \rangle$  when the local chemical potential crosses its critical value. Note the wedding-cake-like form of the profile, characterized by a central doublon-SF, surrounded by a holon-SF ring.

The interpretation of measurements of other observables, as e.g., the momentum distribution in time-of-flight (TOF) measurements, is more involved, since the basis conversion is non-linear,  $\langle \{n'_j\} | b_i^\dagger b_j | \{n_j\} \rangle \sim e^{-iV(t)(n_i - n_j + 1)} \langle \{n'_j\}, m | b_i^\dagger b_j | \{n_j\}, m \rangle$ . However, for the holon- and doublon-SF phases the TOF measurement is almost time-independent for small  $|\mathcal{J}_0(\Omega)|$ . Indeed this weak dependence is in itself a proof of the strong holon or doublon character of the SF. For large  $|\mathcal{J}_0(\Omega)|$  the nonlinear conversion is an issue, and in general measurement results are periodic. Stroboscopic measure-

ments at selected holding times with  $V(t) = 0$  provide, however, a direct conversion between both basis. Interestingly, time averaged measurements over times much larger than the period  $T$  just provide the contribution of  $|n, n \pm 1\rangle \rightarrow |n \pm 1, n\rangle$  processes.

In summary, periodically modulated interactions lead to a rich physics for cold gases in optical lattices, characterized by a nonlinear hopping depending on the number difference at neighboring sites. This hopping can lead to pair superfluid phases, and also to defect-free Mott states, and holon- and doublon superfluids, which may be revealed by abrupt jumps of the in-situ densities in harmonically trapped lattice gases.

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